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EXISTENCE OF Ψ-BOUNDED SOLUTIONS FOR FUZZY DYNAMICAL SYSTEMS ON TIME SCALES

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Abstract:

In this paper, we present a criteria for the existence of Ψ-bounded solutions for linear fuzzy first order systems on time scales. The calculus of timescales is used as a tool to unify both continuous discrete fuzzy systems in a single framework.

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1. Introduction

In this paper, we shall be concerned with first order differential system of the form

$$(x^{\alpha})^{\Delta}(\mathrm{t}) = A(t)\,(x^{\alpha}(\mathrm{t})) + f(t) \qquad (1.1)$$

where (x^{α}) (t), f(t) are in T^k, and A is continuous k × k matrix valued function. We assume that the system (1.1) admits atleast one Ψ-bounded solution for each $\alpha \in [0,1]$. In the year 2020, Kasi Viswanadh, et.al. [5] established existence of Ψ-bounded solutions on fuzzy systems associated with (1.1) on timescales with $\alpha = 0$. We establish existence of Ψ-bounded systems associated with (1.1) on timescales. Qualitative properties of general first order system are established in [3,4]. A new approach to the construction of transition matrix with several applications to control systems are established in [10]. Metrics that are used in this paper are taken from [9]. A note on Ψ-bounded solutions of a system of differential and difference equations are established by Diamandescu A. [11,12]. Three point boundary values problems associated with first order matrix system are established in [1,10]. This paper unifies results established by the authors [1-10] to fuzzy dynamical systems on timescales. It is a well-known fact that mathematical models or equations a physical, a biological or an object oriented design are in most cases either linear difference/differential equations of first order. In recent years, however the investigation of the theory of delta differential equations is attracting the attention of mathematicians to unify both continuous and discrete systems. Moreover the theory of delta differential equations are a lot more richer than the theory of differential/difference equations. The calculus of timescales was basically initiated by Hilger S. [15] in order to create a theory that can unify discrete and continuous analysis. Hence the study of dynamic equations on timescales equations attracted the attention of notable mathematicians as it spreads new light on the discrepancies between continuous and discrete difference systems. The general ideas on basic calculus are taken from Bohner M. and Peterson A. [14]. Further results on first order fuzzy difference equations we refer [13, 16].

This paper is organized as follows. In section 2, we review the basic results on timescale dynamical systems and Ψ-bounded solutions of linear first order systems. Our main results are presented in section 3. In this section, we establish the notions of differential inclusions and establish Ψ-bounded solutions of linear delta differential fuzzy systems. The problem of Ψ-bounded solutions are studied by many authors in recent years [5, 11, 12]. The results established by Viswanadh et.al. [5] unifies both continuous and discrete systems in a single framework

2. Preliminaries

The purpose of this section is to review some useful results, definitions and basic properties on time scales dynamical systems and Ψ-bounded solutions for linear systems which are needful for later discussion. Let T be a time scale, i.e., an arbitrary non-empty closed sub- set of real numbers. Throughout this paper, the time scale T is assumed to be unbounded above and below. In this paper we introduce some notations:

$T^k = (0, \infty) \cap T$,$\nu = \min\{[0, \infty)\cap T\}$. For $x = (x_1, x_2, x_3, \ldots, x_k)T \in T^k$, let

$\|x\| = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_k|\}$ be the norm of x. For any matrix A of order $k \times k$, we define the norm of matrix A [9] as

$$\|A\| = \max_{\|x\|\leq 1} \|Ax\| = |A|$$

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Definition 2.1: A matrix P is said to be a projection if $P^2 = P$. If P is the projection matrix, then I-P is also a Projection. Two such projections, whose sum is I and whose product is zero are said to be supplementary.

Lemma 2.1: If A is delta differentiable at $t \in T^k$, then

$$A(\sigma(t)) = A(t) + \mu(t)A^{\Delta}(t).$$

Definition 2.2: A mapping $f : T \to X$, where X is a Banach space, is said to be rd-continuous if

(i) It is continuous at each right- dense t $\in T$.
(ii) At each left dense point the left side limit f(t) exists.
(iii) Note that if condition (ii) is replaced by f being continuous at each left-dense point, then f is said to be continuous function on T.

Lemma 2.2: A function $F : T^k \to R$ is called an anti-derivative of $f : T^k \to R$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in T^k$ and

$$\int_a^t f(s)\Delta(s) = F(t) - F(a)$$

Lemma 2.3: If f is Δ-differentiable, then f is continuous. Also, if t is right scattered and f is continuous at t, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Theorem 2.2: Assume $f : T \to R$ is a function and let $t \in T^k$, then we have the following:

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If f is right-dense, then f is differentiable at t if the limit

$$\lim_{t \to s} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number, in this case

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$$f^{\Delta}(t) = \lim_{t \to s} \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

Definition 2.3: A function $f : T \to R$ is called regulated provided its right-sided limits exists (finite) at all right-dense points in T and its left-sided limits exist (finite) at all left dense points in T.

Definition 2.4: A function $f : T \to R$ is called rd-continuous provided it is continuous at right-dense points in T and its left-sided limits exists (finite) at left-dense points in T. The set rd-continuous functions $f: T \to R$ will be denoted by

$$C_{rd} = C_{rd}(T) = C_{rd}(T, R).$$

The set of functions $f : T \to R$ that are differentiable and whose derivative is rd- continuous is denoted by

$$C^1_{rd} = C^1_{rd}(T) = C^1_{rd}(T, R).$$

Theorem 2.3: Assume $f : T \to R$:

(i) If f is continuous, then f is rd-continuous
(ii) If f is rd- continuous, then f is regulated.
(iii) The jump operator σ is rd- continuous, then so is f σ.
(iv) Assume f is continuous. If $g : T \to R$ is regulated or rd- continuous, then (fog) has that property too .

Definition 2.5: A function $\phi : T \to R^{k\times k}$ is said to be Ψ-bounded on T if the matrix function $\Psi\phi$ is bounded on T (i.e., there exists $L > 0$ such that $|\Psi(t)\phi(t)| \le L$ for all $t \in T$).

Definition 2.6: A function $\phi : T \to R^{k\times k}$ is said to be Lebesgue Ψ-integrable on T if ϕ is Δ measurable and $\Psi\phi$ is Lebesgue integrable on T.

By a solution of (2.1), we mean an absolutely continuous function satisfying (2.1) for almost all $t \in T$.

Let the vector space R^k be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution y(t) of (2.1) is $\Psi(\sigma(t))$-bounded on T if and only if $x(0) \in X_0$ and $\Psi(\sigma(t))$-bounded on $T_+ = [0, \infty)$ if and only if $x(0) \in X_- \oplus X_0$. Also, let P_-, P_0, P_+ denote the corresponding projection of R^k onto X_-, X_0, X_+ respectively. Note that

$$\Psi(\sigma(t)) = \begin{cases} \Psi(t), if t \in R \\ \Psi(n), if n \in N \end{cases}$$

We now formulate the fuzzy linear system associated with (2.1)

Let $u_i(t) \in T \; i = 1,2,\dots,k$ and define

$$\hat{u}\ (t) = (u_1(t), u_2(t), \dots, u_k(t))$$

$$= \{(\tilde{u}_1(t), \tilde{u}_2(t), \dots, \tilde{u}_k(t))\}$$

$$= \left\{ u_1^{(\alpha)}(t), u_2^{(\alpha)}(t), \dots, u_k^{(\alpha)}(t) \right\}, \text{ for } \alpha \in [0,1]$$

$$= \tilde{u}_i^{\alpha}(t), \alpha \in [0,1]$$

where $u_i^{\alpha}(t)$ be the level set of $u_i(t)$. The fuzzy system associated with (2.1) is

$$\hat{x}_i^{\Delta}(t) = A(t)\, \hat{x}_i(t) + f(t) \tag{2.3}$$

where $t \in T$ (T being the timescale), we consider the following inclusions.

$$\hat{x}(t) = A(t)\, \hat{x}(t) + f(t) \tag{2.4}$$

$$\hat{x}(0) \in \hat{x}_0$$

Let $\hat{x}$ be the solution of (2.4). Then we have

$$\hat{x}(n) = \phi(n)x_0 + \sum_{k=n}^{\infty} \phi(n-j-1)f(t) \tag{2.5}$$

where Φ is a fundamental matrix of $x(n+1) = A(n)\,\hat{x}(0)$ for $t \in N$

For any $t \in T$, the general solution of (2.4) is given by

x(t) = $\Phi(t)\, x_0 \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)$

Both solutions given in (2.5) and (2.6) can be written on timescale dynamical system as

$$x(t) = \Phi(t, t_0)y_0 + \int_{t_0}^{t} \Phi(t, \sigma(s)) f(s)\Delta(s) \qquad (2.7)$$

where $\Phi(t, \sigma(s)) = \Phi(t)\ \Phi^{-1}\sigma(s))$ is a fundamental matrix of

$$\hat{x}(t) = \mathrm{A(t)}\ \hat{x}(t) \qquad (2.8)$$

and the solution belongs to $\hat{x}(t) \in \mathrm{A(t)}\ \hat{x}(t)$

3. Main Results

In this section we present our main result namely Ψ(σ(t))-bounded solutions of the linear system associated with fuzzy linear dynamical systems.

Theorem 3.1: If A is a continuous *(k x k)* matrix on T, then the fuzzy system of non-homogenous linear dynamical system

$$\|\Psi(t)\phi(t)\Psi^{-1}(\sigma(t))\| \leq K \text{ for all t} \geq 0$$

and $\Psi(t)$ be such that

$$\|\Psi(t)\Psi^{-1}(\sigma(s))\| \leq M \text{, for all } 0 \leq s \leq t.$$

Then the linear system (2.8) has atleast one Ψ-bounded solution on T.

Definition 3.1: Let x be a non-empty set. A fuzzy set on A in X is characterized by its membership function $A : x \rightarrow [t_0, t_1]$ and A (n) is interpreted as a degree of the membership of elements n in fuzzy set A, for each $n \in N$.

For $0 \leq \alpha \leq 1$, we define $[y]^\alpha = \{n \in N: y(n) \geq \alpha\}$, it follows that the α-level sets $[y]^\alpha \in E^N$. It is a well known fact that

$$[g(y, \bar{y})]^\alpha = g[y^\alpha, \bar{y}^\alpha]$$

for all $y, \bar{y} \in E^n, 0 \leq \alpha \leq 1$ and $\bar{y}$ is a discrete function.

Definition 3.2: A fuzzy number in parametric form is represented by (u_α^-, u_α^+), where

$u_\alpha^- = min[u]^\alpha$ and $u_\alpha^+ = max[u]^\alpha$, $0 \le \alpha \le 1$.

Definition 3.3: By a fundamental sequence in a Banach space x, we mean a sequence in x whose span in y is dense in x.

In other words every element in x can be approximated by an element in the span. That is, given x ∈ X, we can find a y ∈ Y, such that ||x - y|| is small. This concept is frequently used in this section here after.

Let $\phi^\alpha(t)$ be a fundamental matrix of the linear fuzzy homogenous system for any fixed $\alpha \in [0,1]$. We denote here after

$$\phi^\alpha(t) = \hat{\phi}(t)$$

The membership function of the fuzzy set on T close to 0 can be defined as

$$A(\sigma(t)) = \frac{1}{1+(\sigma(t))^2}.$$

where $\sigma(t)$ = t if $t \in R$ and $\sigma(t)$ = n if $t \in N$

The membership function of the fuzzy set on the timescale T close to one can be defined as

$$B(\sigma(t)) = \exp[-\beta(\sigma(t) - 1)]$$

For instance, the number 3 is assigned a grade 0.035, the number 1 a grade of 0.5 and the number 0 a grade of 1 on real set.

Theorem 3.1: Let A be a *(k x k)* matrix on T, then the fuzzy system of non-homogenous linear dynamical system

$$\hat{x}^\Delta(t) = A(t)\, \hat{x}_i(t) + f(t) \qquad (3.1)$$

has at least one Ψ-bounded solution for each $\alpha \in [0,1]$ for every Lebesgue Ψ- delta integrable function $f : T \to T^k$ if and only if there exists a positive constant L > 0 such that

$$|\Psi(t)\, \hat{\Phi}(t) P_-\, \hat{\Phi}^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))| \le L \text{ for } t > 0,\ \sigma(s) \le 0$$

$$|\Psi(t)\hat{\Phi}(t)(P_0 + P_-)\hat{\Phi}^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))| \le L \ for \ t > 0, \sigma(s) > 0, \sigma(s) < t$$

$$|\Psi(t)\hat{\Phi}(t)P_+\hat{\Phi}^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))| \le L \ for \ t > 0, \sigma(s) > 0, \sigma(s) \ge t$$

$$|\Psi(t)\hat{\Phi}(t)P_-\hat{\Phi}^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))| \le L \ for \ t \le 0, \sigma(s) < t$$

$$|\Psi(t)\hat{\Phi}(t)(P_0 + P_+)\hat{\Phi}^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))| \le L \ for \ t \le 0, \sigma(s) \ge t, \sigma(s) < 0$$

$$|\Psi(t)\hat{\Phi}(t)P_+\hat{\Phi}^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))| \le L \quad for \ t \le 0, \sigma(s) \ge t, \sigma(s) \ge 0.$$

Proof: Suppose the system (3.1) has atleast one Ψ-bounded solution on T for each $\alpha \in [0,1]$ for every Lebesgue Ψ- delta integrable function $f : \mathrm{T} \to \mathrm{T}^k$, we now define

(i) C_Ψ : the Banach space of all Ψ-bounded continuous function $\hat{x} : \mathrm{T} \to \mathrm{T}^k$ with the norm $||\hat{x}|| = \sup_{t \in T} |\Psi(t)x(t)|$.

(ii) B is the set of all Lebesgue Ψ-Δ integrable function $x : \mathrm{T} \to \mathrm{T}^k$ such that
$||x||_B = \int_{-\infty}^{\infty} |\Psi(t)x(t)| t$, and

(iii) D : the set of all functions $x : \mathrm{T} \to \mathrm{T}^k$ which are absolutely continuous on all subinterval J ⊂ T such that Ψ is bounded on T,
$\hat{x}(0) = X_- \oplus X_+$ and $\hat{x}^\Delta = A\hat{x} \in B$

It can easily be verified that D is a vector space and

$$\hat{x} \to ||\hat{x}||_D = ||\hat{x}||_{C_\Psi} + ||\hat{x}^\Delta - A\hat{x}||_B$$

is a norm on D.

Step1: We first claim that (D, $||.||_D$) is a Banach space. Let

$\left\{ \hat{x}_n \right\}$ $n \in N$ be a fundamental sequence in D. Then $\left\{ \hat{x}_n \right\}$ is a fundamental sequence in C_Ψ. Therefore there exists a continuous Ψ-bounded function $\hat{x} : \mathrm{T} \to \mathrm{T}^k$ such that

$$\lim_{n\to\infty} \Psi(t)x(t) = \Psi(t)\hat{x}(t)$$

Hence $\Psi(t)\hat{x}(t)$ is uniformly continuous. Then $\hat{x}(0) \in X_- \oplus X_+$.

Let f_n be a sequence such that

$f_n(t) = \hat{x}_n(t) - A(t)\hat{x}_n(t)$ is also a fundamental sequence in B. Then there exists $f \in B$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \Psi(t)(f_n(t) - f(t))\Delta t = 0$$

For a fixed $t \in T$, we have

$$\hat{x}(t) - \hat{x}(0) = \lim_{n\to\infty} \hat{x}_n(t) - \hat{x}_n(0)$$

$$= \lim_{n\to\infty} \int_0^t \hat{x}_n^\Delta(s) - A(s)\hat{x}_n(s) + A(s)\hat{x}_n(s)$$

$$= \lim_{n\to\infty} \int_0^t \Psi^{-1}(s)\Psi(s)f_n(s) + A(s)\hat{x}_n(s)$$

$$= \lim_{n\to\infty} \int_0^t [f(s) + A(s)\hat{x}_n(s]\Delta s$$

It follows that $\hat{x}^\Delta - A\hat{x}$ and x is absolutely continuous on all interval $J \subset T$. Thus $\hat{x} \in D$ and

$\lim_{n\to\infty} x_n(t) = x(t)$.

Step 2: There exists a $\mathbb{K} > 0$ such that for every $f \in B$ and for corresponding solution $\hat{x} \in D$, we have

$$\sup_{t\in T} \|\Psi(t)\hat{x}(t)\| \leq \mathbb{K}, \int_{-\infty}^{\infty} \|\Psi(t)(f(t)\|\Delta t$$

Let $T: D \to B$ such that $T\hat{x} = \hat{x}_\delta - A\hat{x}$.

Then T is linear and bounded with $\|T\| \leq 1$. Let $T\hat{x} = 0$, then $\hat{x}^\delta - A\hat{x} = 0$ and $x \in D$. Hence, $\hat{x}$ is ψ-bounded solution on T of (3.1). Then $\hat{x}(0)$ belongs to $x_0 \cap (X_- \oplus X_+) = 0$. Thus x = 0, so that T is one-one.

Moreover T is onto. From a fundamental result on Banach space T^{-1} is also a bounded linear operator. Hence (3.2) holds.

Step 3: Let $\theta_1 < 0 < \theta_2$ be arbitrary fixed points and let $f: T \to T^k$ be a function in B which vanishes on $(-\infty, \theta_1) \cup (\theta_2, \infty)$.

It can easily be shown that the function $\hat{x}: T \to T^k$ defined by

$$\hat{x}(t) = \begin{cases} -\int_{\theta_1}^{0} \phi(t)P_0\phi^{-1}(\sigma(s))f(s)ds - \int_{\theta_1}^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s, t < \theta_1 \\ -\int_{\theta_1}^{t} \phi(t)P_-\phi^{-1}(\sigma(s))f(s)\Delta s + \int_0^t \phi(t)P_0\phi^{-1}(\sigma(s))f(s)\Delta s \\ -\int_{\theta_1}^{\theta_2} \phi(t)P_-\phi^{-1}(\sigma(s))f(s)\Delta s, , \theta_1 \le t \le \theta_2 \\ \int_0^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s, t > \theta_2 \end{cases}.$$

is the solution in D of the system (3.1). Now that

$$\begin{aligned} G(t,\sigma(s)) &= \phi(t)P_-\phi^{-1}(\sigma(s)), \sigma(s) \le 0 \le t \\ &= \phi(t)(P_0 + P_-\phi^{-1}(\sigma(s)), 0 < \sigma(s) < t \\ &= \phi(t)P_+\phi^{-1}(\sigma(s)), 0 < t \le \sigma(s) \\ &= \phi(t)P_-\phi^{-1}(\sigma(s)), \sigma(s) < t \le 0 \\ &= \phi(t)(P_0 + P_-\phi^{-1}(\sigma(s)), t \le \sigma(s) < 0 \\ &= \phi(t)P_+\phi^{-1}(\sigma(s)), t \le 0 \le \sigma(s) \end{aligned}$$

then $x(t) = \int_{\theta_1}^{\theta_2} G(t,\sigma(s))f(s)\Delta s, t \in T$
$=\hat{x}(t)$

$G(t,\sigma(s))$ is continuous on T^2 except on the line $t = \sigma(s)$, where it has a jump discontinuity.

For $t < \theta_1$, we have

$$\int_{\theta_1}^{\theta_2} G(t,\sigma(s))f(s)\Delta s = \int_{\theta_1}^{0} \phi(t)(P_0 + P_-\phi^{-1}(\sigma(s))f(s)\Delta s - \int_0^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s$$

For $t \in [\theta_1, 0]$ and $t \in (0, \theta_2)$ and for $t > \theta_2$ it can similarly be shown that the corresponding integrals are equal to $\hat{x}(t)$.

Now, the inequality (3.2) becomes

$$\sup_{t\in T}\|\Psi(t)\| \int_{\theta_1}^{\theta_2} \|G(t,\sigma(s))f(s)\Delta s\| \le \mathbb{K}, \int_{\theta_1}^{\theta_2} \Psi(t)(f(t)\Delta t$$

For a fixed point $\sigma(s) \in T, \delta > 0$ and $\xi \in T^k$(ξ aribitrary), let f be defined by

$$f(t) = \begin{cases} \Psi^{-1}(t)\xi, \sigma(s) \le t \le \sigma(s) + \delta \\ 0, otherwise \end{cases}. \quad (3.3)$$

clearly $f \in B, \|f\|_B = \delta\|\xi\|$. Hence (3.3) becomes

$$\int_s^{s+\delta} \|\Psi(t)G(t,\sigma(s))|Psi^{-1}(\sigma(s))\xi\| \le \mathbb{K}\delta\|\xi\|$$

for all $t \in T$.
Dividing by s and taking the limit as $\delta \in 0$, we get

$$\|\Psi(t)G(t,\sigma(s))|Psi^{-1}(\sigma(s))\xi\| \leq \mathbb{K}\|\xi\|, \text{for all } t \in T, \xi \in T^k$$

Hence, $\|\Psi(t)G(t,\sigma(s))|Psi^{-1}(\sigma(s))\xi\| \leq \mathbb{K}$, which is equivalent to (3.1). Since $\alpha \in [0,1]$ holds for all α the result follows.

The other part is straight forward.
Theorem 3.3: If the homogenous equation $\hat{x}^{\Delta}(t) = A(t)\hat{x}(t)$ has no nontrivial Ψ-bounded solution on T, then (3.1) has a unique Ψ-bounded solution on T for every Lebesgue ψ-delta integrable function $f: T \to T^k$ if and only if there exists a $\mathbb{K} > 0$, such that

(i) $\|\Psi(t)\phi(t)P_{-}\phi^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))\| \leq k\sigma(s)$, for $-\infty < \sigma(s) < t < \infty$
(ii) $\|\Psi(t)\phi(t)P_{+}\phi^{-1}(\sigma(s))\Psi^{-1}(\sigma(s))\| \leq k\sigma(s)$, for $-\infty < t \leq \sigma(s) < \infty$

In this case we take $P_0 = 0$ in the above theorem.

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